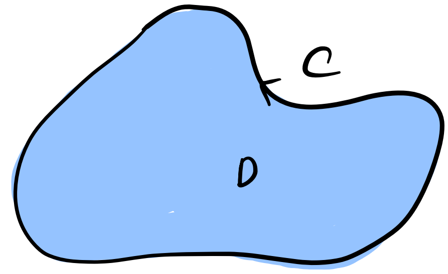
Review: Green theorem is a tool to compute line integral when the normal method of parametrizing is too complicated.



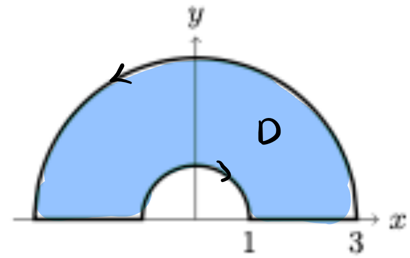
positive orientation

$\vec{F} = (P, Q)$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA .$$

(2nd type line integral)

$\boxed{\text{Example 1}}$. Find the word done by $\vec{F} = (4x - 2y, 2x - 4y)$ once counter clock-wise around



Using Green's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial(2x-4y)}{\partial x} - \frac{\partial(4x-2y)}{\partial y} \right) dA$$

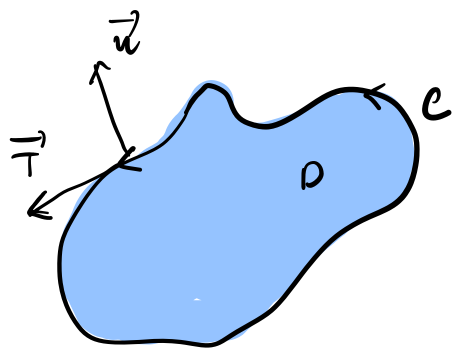
$$= \iint_D (2 - (-2))\, dA = 4 \text{ area}(D)$$

$2D = (\text{circle of radius } 3) - (\text{circle of radius } 1)$

thus

$$4 \text{ area}(D) = 4 \cdot \frac{1}{2} \cdot (\pi \cdot 3^2 - \pi \cdot 1^2) = \boxed{16\pi}$$

Another form of Green's theorem.



When travelling on the curve C, parametrized by $r(t)$, then $T(t) = r'(t)$ is a tangent

$n(t)$: normal at the point (should take unit normal)

Green's theorem (standard)

- Usual line integral (tangent component) $\oint_C \vec{F} \cdot \vec{r}\, dr = \oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$
- The normal component $\oint \vec{F} \cdot \vec{n}\, dr = \iint \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$

(flux) $\quad {}_C \quad \uparrow \quad {}_D \left(\partial x \quad \partial y \right)$

divergence

New form of Green's theorem
(only when computing the normal component)

How to find the normal vector

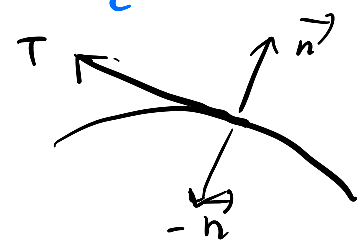
(if the curve is not closed
→ cannot use Green's theorem
thus, we have to find $\vec{r}(t)$, $\vec{n}(t)$
to compute $\oint_C \vec{F} \cdot \vec{n}\, dr$ (flux))

If $r(t) = (x(t), y(t))$

$T(t) = r'(t) = (x'(t), y'(t))$

$\uparrow$
tangent

↳ then $\vec{n}(t) = \begin{cases} (-y', x') \\ (y', -x') \end{cases}$ or



as long as $\vec{n} \cdot \vec{r}\,' = 0$

The choice of $\vec{n}$ depends on the problem.

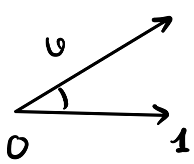
$\boxed{\text{Example 2}}$. Consider $\vec{r}(t) = (1+t, 3-t^2)$ for $t \in [0,2]$

Find an equation for the normal that points to the right.
(unit normal)

$r'(t) = (1, -2t) \Rightarrow \vec{n}(t) = \begin{cases} (2t, 1) \\ (-2t, -1) \end{cases}$ or

Check : which one points to the right?

$\Rightarrow$ check : $\vec{n} \cdot (0,1) \geq 0$



$(2t, 1) \cdot (0,1) = 1 > 0 \rightarrow$ pick.

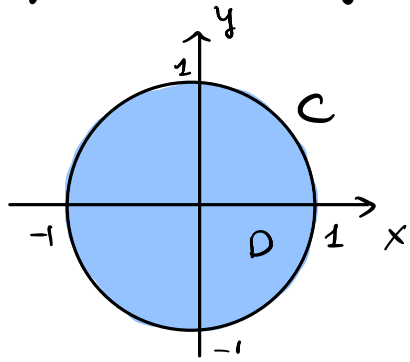
$-(2t, 1) \cdot (0,1) = -1 < 0$

thus the unit normal that points to the right is

$$\vec{n} = \frac{(2t, 1)}{\sqrt{4t^2+1}}.$$

$\boxed{\text{Example 3}}$. Calculating the outward flux of $\vec{F} = (x+3, xy-5)$ across C : $x^2+y^2 = 1$

$$\oint_C \vec{F} \cdot \vec{n} \, dr$$



Proof 1 . Green's theorem : (flux form)

$$\oint_C \vec{F} \cdot \vec{n} \, dr = \iint_D \left(\frac{\partial (x+3)}{\partial x} + \frac{\partial (xy-5)}{\partial y} \right) dA$$

$$= \iint_D (1+x) \, dA \rightarrow \text{simple double integral}$$

$$= \int_0^{2\pi} \int_0^1 (1 + r\cos\theta) \underbrace{r}_{\text{Jacobian}} \, dr \, d\theta$$

Proof 2 . Using the definition, compute $\vec{n}$.

$$r(t) = (\cos t, \sin t) \Rightarrow r'(t) = (-\sin t, \cos t)$$

$$\Rightarrow n(t) = \begin{cases} (\cos t, \sin t) & \rightarrow \text{outward} \\ \text{or} \\ (-\cos t, -\sin t) & \rightarrow \text{inward} \end{cases}$$

$$\oint_C \vec{F} \cdot \vec{n} \, dr = \int_0^{2\pi} (\cos t + 3, \cos t \sin t - 5) \cdot (\cos t, \sin t) \, dt$$

$$= \int_0^{2\pi} \left(\cos^2 t + 3\cos t + \cos t \sin^2 t - 5\sin t \right) dt$$

$\hookrightarrow$ more complicated integral.

Example 4. Consider the curve $C : x^2+y^2=9$

Calculate the outward flux of $\vec{F} = (x^2y, y)$ across C.

$$\oint_C \vec{F}\cdot\vec{n}\, ds$$

Proof 1. Green's theorem: (flux form)

$$\iint_D \left(\frac{\partial(x^2y)}{\partial x} + \frac{\partial(y)}{\partial y} \right) dA = \iint_D (2xy+1)\, dA$$

$$= \text{area of } D + 2\iint_D xy\, dA$$

Jacobian.

$$= \pi\cdot 3^2 + 2\int_0^{2\pi}\int_0^3 r^2\cos\theta\sin\theta \cdot r\, dr\, d\theta$$

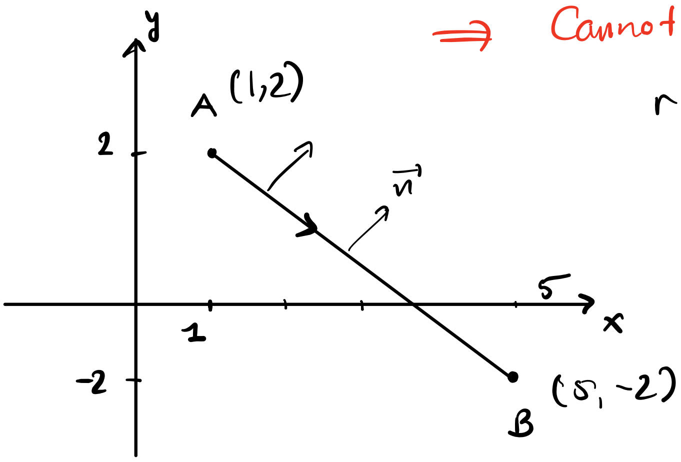
Proof 2. $\vec{n} = (\cos t, \sin t)$ (unit normal), $r(t) = (3\cos t, 3\sin t)$

$$\int_0^{2\pi} (9\cos^2 t\cdot 3\sin t,\ 3\sin t)\cdot(\cos t, \sin t)\, dt$$

$\hookrightarrow$ lengthy.

Example 5. Calculate the upward flux of $\vec{F} = (3x, 2y)$ across C: the line segment from $(1,2)$ to $(5,-2)$

$\Rightarrow$ Cannot use Green's theorem as C is not closed.



$r(t) = tB + (1-t)A \qquad t\in[0,1]$

$= t(5,-2) + (1-t)(1,2)$

$= (4t+1, -4t)$

thus $r'(t) = (4,-4)$

$\Rightarrow \vec{n}(t) = (4,4)$ or $(-4,-4)$

pick this, as upward

make $\vec{n}(t) = \frac{(1,1)}{\sqrt{2}}$ as unit normal

$$\oint_C \vec{F} \cdot \vec{n} \, dr = \int_0^1 \Big(3(4t+1),\ 2(-4t)\Big) \cdot \frac{(1,1)}{\sqrt{2}} \, dt$$

$$= \frac{1}{\sqrt{2}} \int_0^1 \left(12t + 3 - 8t \right) dt$$

$$= \frac{1}{\sqrt{2}} \int_0^1 (4t+3) \, dt$$

$$= \frac{1}{\sqrt{2}} \left(4 \cdot \frac{1}{2} + 3 \right) = \frac{1}{\sqrt{2}} (5) = \frac{5}{\sqrt{2}}$$